A Classical Generalized Variational Principle For Pseudo-Static Thermoelasticity Of Piezoelectric Materials

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Abstract: It is very difficult to establish a classical variational principle (not Gurtin-typenot involving convolutions)for thermopiezoelasticity, however, the semi-inverse method proposed by He appears to be one of the best and most convenient ways to establish variational principles for the physical problems. By such method, a classical generalized variational principle with 9 kinds of independent variations has been established for the pseudo-static thermoelasticity of piezoelectric materials.

Keywords: *Variational Theory, Semi-Inverse Method, thermopiezoelectricity*

INTRODUCTION

Recent interest in piezoelectric materials stems from their potential applications in intelligent structural systems. A comprehensive list of works in this area may be found in[1~4] and the references cited thereby. The rapid development of computer science and the finite element applications reveals the importance of searching for a classical variational principle for the thermopiezoelectricity, which is the theoretical basis of the finite element methods[5] and meshfree methods[6].

A GENERALIZED VARIATIONAL PRINCIPLE

Though it is easy to establish a Gurtin-type functional(involving convolutions), it is very difficult to construct a classical variational model due to the strongly coupled constitutive relations and the terms of the first-order time-derivatives involving in the heat conduction equation. As the author knows, there exist no such classical variational models for the thermopiezoelectricity, the semi-inverse method[6~9] that we are proposing appears to be one of the best and most convenient ways to establish variational principles for the physical problems. By such method we obtained following generalized variational principle with 9 kinds of independent variations (stress σ_{ij}, strain γ_{ij}, displacement u_i, temperature θ, heat flux q_i, electric displacement D_i, electric field E_i, electric potential Φ and entropy S)

$$J(\sigma_{ij},\gamma_{ij},u_i,\theta,q_i,D_i,E_i,\Phi,S)=\int_{t^{(n-1)}}^{t^{(n)}}\int L\,\mathrm{d}V\,\mathrm{d}t+IB$$

where

$$\begin{aligned}L=&\sigma_{ij}\gamma_{ij}-\tfrac{1}{2}\sigma_{ij}(u_{i,j}+u_{j,i})\\&+\gamma_{ij}(-\tfrac{1}{2}a_{ijkl}\gamma_{kl}+e_{mij}E_m+b_{ij}\theta)+f_iu_i\\&+\theta(\tfrac{1}{2}c\theta_0\theta+c_iE_i-t'q_{i,i}-\alpha)\\&+\tfrac{1}{2}(K_{ij}\tau+t')q_iq_j-\beta q_i-E_iD_i+\tfrac{1}{2}E_i\varepsilon_{ij}E_j+D_i\Phi_{,i}\\&+\lambda(\rho S-c\theta-b_{ij}\gamma_{ij}-c_iE_i)^2,\end{aligned}$$

$$\begin{aligned}IB=&\int_{A_1}\sigma_{ij}n_j(u_i-\bar{u}_i)\,\mathrm{d}A+\int_{A_2}\bar{p}_iu_i\,\mathrm{d}A\\&-\int_{A_3}(\Phi-\bar{\Phi})D_in_i\,\mathrm{d}A-\int_{A_4}\bar{D}_n\Phi\,\mathrm{d}A\\&+\int_{A_5}t'q_in_i\bar{\theta}\,\mathrm{d}A+\int_{A_6}t'\theta(q_in_i-\bar{q}_n)\,\mathrm{d}A,\end{aligned}$$

where $t'=t-t^{(n-1)}$, $t\in[t^{(n-1)},t^{(n)}]$, λ is a nonzero constant, α and β are written in the forms

$$\alpha=c\theta_0\theta^{(n-1)}+b_{ij}\gamma_{ij}^{(n-1)}+c_iE_i^{(n-1)}+t'\rho Q \text{ and } \beta=K_{ij}\tau q_i^{(n-1)}$$

$A_1+A_2=A_3+A_4=A_5+A_6+A_7=A$ covers the total boundary surface.

Making the above functional stationary, we obtain following Euler equations

$$\delta u_i:\ \sigma_{ij,j}+f_i=0 \tag{1}$$

$$\delta S:\ \rho S=c\theta+b_{ij}\gamma_{ij}+c_iE_i \tag{2}$$

$$\delta\gamma_{ij}:\ \sigma_{ij}-a_{ijkl}\gamma_{kl}+e_{mij}E_m+b_{ij}\theta-2\lambda b_{ij}(\rho S-c\theta-b_{mn}\gamma_{mn}-c_mE_m)=0 \tag{3}$$

$$\delta E_m:\ e_{mij}\gamma_{ij}+c_m\theta-D_m+\varepsilon_{mj}E_j-2\lambda c_m(\rho S-c\theta-b_{ij}\gamma_{ij}-c_iE_i)=0 \tag{4}$$

$$\delta\sigma_{ij}:\ \gamma_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \tag{5}$$

$$\delta\Phi: \quad D_{i,i} = 0 \tag{6}$$

$$\delta D_i: \quad E_i = \Phi_{,i} \tag{7}$$

$$\delta\theta: c\theta_0\theta + b_{ij}\gamma_{ij} + c_i E_i - t' q_{i,i} - c\theta_0\theta^{(n-1)} - b_{ij}\gamma_{ij}^{(n-1)} - c_i E_i^{(n-1)} - t'\rho Q$$
$$-2\lambda c(\rho S - c\theta - b_{ij}\gamma_{ij} - c_i E_i) \tag{8}$$

$$\delta q_i: t'\theta_{,i} + (K_{ij}\tau + t')q_i - K_{ij}q_i^{(n-1)} = 0 \tag{9}$$

and following boundary conditions

$$u_i = \overline{u}_i \quad (\text{on} \quad A_1) \tag{10A}$$
$$\sigma_{ij}n_j = \overline{p}_i \quad (\text{on} \quad A_2) \tag{10B}$$
$$\Phi = \overline{\Phi} \quad (\text{on} \quad A_3) \tag{10C}$$
$$D_i n_i = \overline{D}_n \quad (\text{on} \quad A_4) \tag{10D}$$
$$\theta = \overline{\theta} \quad (\text{on} \quad A_5) \tag{10E}$$
$$q_i n_i = \overline{q}_n \quad (\text{on } A_6) \tag{10F}$$

The equations (3) ,(4), (8) and (9), in view of the equation (2), can be re-written down as follows

$$\sigma_{ij} = a_{ijkl}\gamma_{kl} - e_{mij}E_m - b_{ij}\theta \tag{2'}$$
$$D_m = e_{mij}\gamma_{ij} + c_m\theta + \varepsilon_{mj}E_j \tag{3'}$$

$$c\theta_0 \frac{\theta - \theta^{(n-1)}}{t - t^{(n-1)}} + b_{ij}\frac{\gamma_{ij} - \gamma_{ij}^{(n-1)}}{t - t^{(n-1)}} + c_i \frac{E_i - E_i^{(n-1)}}{t - t^{(n-1)}} = q_{i,i} + \rho Q \tag{8'}$$

$$\theta_{,i} = -K_{ij}(\tau \frac{q_i - q_i^{(n-1)}}{t - t^{(n-1)}} + q_i) \tag{9'}$$

When $t \to t^{(n-1)}$, we have

$$c\theta_0 \frac{\partial\theta}{\partial t} + b_{ij}\frac{\partial\gamma_{ij}}{\partial t} + c_i \frac{\partial E_i}{\partial t} = q_{i,i} + \rho Q \tag{8''}$$

$$\theta_{,i} = -K_{ij}(\tau \frac{\partial q_i}{\partial t} + q_i) \tag{9''}$$

or

$$\tau \frac{\partial q_i}{\partial t} + q_i = -k_{ij}\theta_{,j} \tag{9'''}$$

where $\theta = T - \theta_0$, T is the temperature and θ_0 is the initial temperature , Q is the strength of the internal heat source, K_{ij} is the inverse of k_{ij}.

The obtained Euler equations satisfy all the field equations and boundary conditions of the thermoelasticity of piezoelectric materials.

CONCLUSION

Hereby we obtain a variational principle for the discussed problem, which might find some potential applications.

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