Stability of Fully Developed Turbulent Channel Flow

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Abstract: In our earlier work (Sen and Veeravalli, 1998a,b) we have shown the relevance of stability theory in understanding the dynamics of turbulence in fully developed wall-bounded flows. Here we wish to extend the work presented in Sen and Veeravalli (1998b) by using a composite model for calculating both inner and (damped) outerstability modes, for a fully developed turbulent channel. Some comparisons with the experiements of Hussain and Reynolds (1972) are also presented.

Keywords: *Organised disturbances, stability, turbulent channel flow, and anisotropic model for stress tensor*

INTRODUCTION

In Sen and Veeravalli (1998a), hereinafter referred to as S and V, we had examined the relevance of hydrodynamic stability theory to understanding fully developed turbulence in wall bounded flows. While the role of hydrodynamic stability theory in predicting the dominant instability mechanism in free shear flows is well known and well studied (see for example Gaster et al., 1985 Liu, 1988 and Roshko, 1992), its usefulness in wall bounded flows has remained in question. Here we will not go into the history of this problem starting with the pioneering works of Malkus (1956), Reynolds and Tiederman (1967), Reynolds and Hussain (1972), Hussain and Reynolds (1972) etc. The interested reader is referred to S and V for these details.

S and V mainly concerned itself with the stability of turbulent boundary layers. In Sen and Veeravalli (1998b), hereinafter referred to as S and V2, we had considered the specifics of turbulent channel flow. We had shown that with the help a generalised (anisotropic) eddy viscosity model, along the lines of Pope (1975), some of the key features of wall turbulence could be captured by studying the behaviour of organised disturbances in fully developed turbulent flow. In order to enable comparisons with experiments, two separate versions of the anisotropy model had to be used to obtain inner and outer modes. Here we wish to present a composite model that enables us to calculate both inner and outer modes. Some comparisons with the experiments of Hussain and Reynolds (1972) are also presented.

THEORY

In the discussion to follow the instantaneous velocity vector is u_i and the pressure is p. All physical quantities have been normalised by the average velocity, V, the channel half width, H, the kinematic viscosity, ν, and density, ρ. The characteristic Reynolds number R is defined as VH/ν. For some of the discussions to follow it is useful to non-dimensionalise using the inner velocity scale, ν_*, which is the wall friction velocity, and the inner length scale, ν/ν_*. All quantities non-dimensionalised by inner variables are denoted by the superscript (+). The mean velocity is in the x direction, y is normal to the wall, and the mean flow is independent of the x and z directions. (x_1, x_2, x_3) and (x, y, z) will be used interchangably and so also (u_1, u_2, u_3) and (u, v, w). Figure 1 shows the definition sketch of the flow.

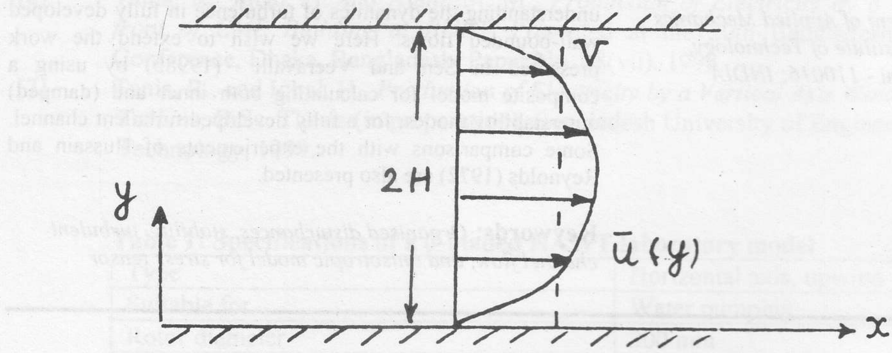

Figure 1: Sketch of the flow

We use the standard Reynolds decomposition:

$$u_i = \bar{u}_i + u_i' \quad ; \quad p = \bar{p} + p' \tag{1}$$

where, $\bar{u}_i$, $\bar{p}$ are respectively the time averaged velocity and pressure, and u_i', p' are the turbulent fluctuations.

We now superimpose an organised (solenoidal) disturbance, $\tilde{u}_i$, $\tilde{p}$ (with zero mean), then the instantaneous velocity and pressure are respectively:

$$u_i = \bar{u}_i + \tilde{u}_i + u_i' \quad ; \quad p = \bar{p} + \tilde{p} + p' \tag{2}$$

The time averages of u_i and p remain the same but the ensemble (phase locked) averages are different. We denote ensemble averages by pointed brackets <>. Then,

$$\langle u_i \rangle = \bar{u}_i + \tilde{u}_i \quad ; \quad \langle p \rangle = \bar{p} + \tilde{p} \tag{3}$$

We assume that the organised disturbance is small in the following sense,

$$\left| \langle \tilde{u}_i \tilde{u}_j \rangle \right| << \left| \langle u_i' u_j' \rangle \right| \tag{4}$$

We now introduce (twice) the rate of strain rate tensor and (twice) the vorticity tensor, respectively s_{ij} and ω_{ij} as follows :

$$s_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \; ; \; \omega_{ij} = \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \tag{5a, b}$$

The mean and ensemble averages are $\bar{s}_{ij}$, $\bar{\omega}_{ij}$ and $\langle s_{ij} \rangle$ and $\langle \omega_{ij} \rangle$ respectively.

As in S and V, following Pope (1975), we model the time averaged Reynolds stress tensor as follows:

$$-\overline{u_i' u_j'} = -\frac{2}{3} k \delta_{ij} + \varepsilon \bar{s}_{ij} - \varepsilon C \left(\frac{k}{\varepsilon_d} \right) \left[\frac{1}{2} \left[\bar{\omega}_{ik} \bar{s}_{kj} - \bar{s}_{ik} \bar{\omega}_{kj} \right] \right] ; \tag{6}$$

Here, ε is the eddy viscosity defined as $-\overline{u'v'}/(d\bar{u}/dy)$; k is the turbulent kinetic energy; ε_d is the dissipation rate of the turbulent kinetic energy and C is a constant. From the form of the above equation we can define an anisotropy parameter λ as $C(k/\varepsilon_d)d\bar{u}/dy'$. Thus the standard isotropic form of the eddy viscosity model is recovered when λ is zero.

The ensemble average of the Reynolds stress similarly is modelled as:

$$-\langle u_i' u_j' \rangle = -\frac{2}{3} k \delta_{ij} + \varepsilon \langle \delta_{ij} \rangle - \varepsilon \left(\frac{\lambda}{(\bar{u})'} \right) \left[\frac{1}{2} \left[\langle \omega_{ik} \rangle \langle s_{kj} \rangle - \langle s_{ik} \rangle \langle \omega_{kj} \rangle \right] \right] \tag{7}$$

Here, $\bar{u}'$, is the mean velocity gradient.

Since the organised disturbance is solenoidal we can define a stream function ψ such that $\tilde{u} = \partial\psi/\partial y$ and $\tilde{v} = -\partial\psi/\partial x$ and if we assume normal modes then ψ may be written as,

$$\psi = \phi(y) e^{i\alpha(x-ct)} \tag{8}$$

where, ϕ is the eigenfunction, α is the spatial wavenumber and $c = c_r + ic_i$ is the complex wave speed.

u_i obeys the Navier-Stokes and continuity equations and from these the evolution equations for $\langle u_i \rangle$ may be obtained by ensemble averaging. If we subtract the equations for $\langle u_i \rangle$ from those for u_i, we obtain the equations governing the evolution of $\tilde{u}$. Then using the model for the Reynolds stresses as above we obtain the following extended Orr-Sommerfeld equation as in S and V,

$$
\begin{aligned}
& i\alpha\left[(\bar{u}-c)\left(\phi''-\alpha^2\phi\right)-\bar{u}''\phi\right]-1/R\left[\phi''''-2\alpha^2\phi''+\alpha^4\phi\right] \\
& \quad -1/R\left[\varepsilon\left\{\phi''''-2\alpha^2\phi''+\alpha^4\phi\right\}+2\varepsilon'\left\{\phi'''-\alpha^2\phi'\right\}+\varepsilon''\left\{\phi''+\alpha^2\phi\right\}\right] \\
& \quad -\frac{\lambda\varepsilon}{R}\left[-2i\alpha\phi'''+2i\alpha^3\phi'\right]-\frac{2i\alpha\phi'}{R}\left[\lambda\varepsilon''+2\lambda'\varepsilon'+\lambda''\varepsilon\right]=0 \qquad (9)
\end{aligned}
$$

Primes (′) denote differentiation with respect to y. In (9), the first group of terms in square brackets corresponds to the Rayleigh equation; the first two groups of terms in square brackets correspond to the classical Orr-Sommerfeld equation; and, the remaining terms constitute the modifications in the classical Orr-Sommerfeld equation.

The reader is referred to S and V for a more detailed derivation and for a discussion of the subtleties involved in the model.

The boundary conditions at the wall y=0 are,

$$\phi = 0\ ;\quad \phi' = 0 \qquad (10)$$

And at the channel centre y=1 are,

$$\phi' = 0\ ;\quad \phi''' = 0 \qquad \text{anti-symmetric mode}; \qquad (11a)$$

$$\phi = 0\ ;\quad \phi'' = 0 \qquad \text{symmetric mode}; \qquad (11b)$$

In order to compute the eigenfunction expressions for ε and λ as functions of y are required. Reynolds and Tiederman (1967) report an analytical expression for ε which matches well with experimental measurements. The same expression is used here.

In S and V, two expressions were given for λ, following a detailed discussion of how it is to be obtained from experimental data (Klebenoff, 1954) and numerical simulations (Spalart 1988). The first expression was given in inner variables and matches reasonably well with experimental data in the wall region, while the second expression in outer variables was smoother and easier to use from a computational viewpoint. However, both expressions yield a value of 3 for λ at the channel centre. Since the turbulence at the centre of the channel is nearly isotropic, λ should approach a value of zero at the centre. This is not a serious error if we are only interested in wall modes, however, if outer modes are to be calculated, also then it is important to enforce the proper value at the channel centre. Thus, for the results presented here, the expression for λ is as follows:

$$\lambda = f\left(y^{+}\right)g(y)$$

where,

$$f\left(y^{+}\right)=10.5\left[3/10.5+\left(\frac{1-3/10.5}{2}\right)\left\{1-\operatorname{erf}\left(\frac{y_{+}-18.0}{7.5}\right)\right\}\right]; \tag{12a}$$

and,

$$g(y)=0.5\left[1-\operatorname{erf}\left(\frac{y-0.25}{0.075}\right)\right] \tag{12b}$$

Figure 2 shows the resulting variation of λ with y at a Reynolds number of 5000. Also shown in the figure is λ estimated from the experimental data of Klebanoff(1955). The rather high value of $\lambda \approx 25$ shown by the experimental data near the wall is not critical to the model and we find that good results are obtained by allowing λ to start from a value of 10.5.

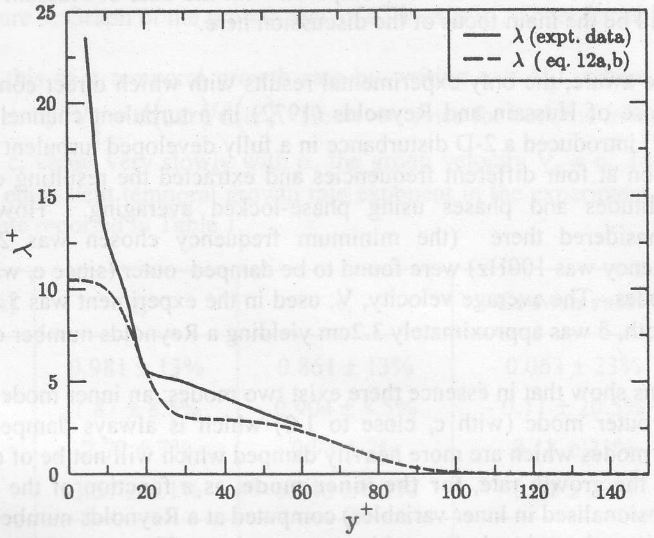

Figure 2: Graph of Anisotropy Function $\lambda^{+}(=\lambda)$ versus y^{+} standardised in terms of inner variables. Based on experimental data of Klebanoff, and from eq.(12a,b)

Finally, an analytically continuous and realistic turbulent mean velocity profile is required. This is obtained by the same procedure as outlined in Reynolds and Tiederman (1967), S and V2.

RESULTS

The details of the numerical scheme used to compute the eigenvalues are given in Sen and Vashist (1989) and Sen et al., (1985). The calculations were done for symmetric modes only as they are more unstable. A value for α was chosen and the complex wave speed, c, was computed as the eigen value. A positive value for c_i, thus, indicates instability.

In S and V2, the results for three different Reynolds numbers (5000, 11000 and 17000) were presented. In each case a wide range of unstable wave numbers was obtained with the wave speed $c_r \approx 0.3$, which corresponds to the inner mode. Comparisons with the results for the boundary layer showed that the results for the channel flow and the boundary layer were nearly indistinguishable at all the three Reynolds numbers, when plotted in inner variables. From physical considerations, this is expected as the wall region dynamics is nearly independent of the outer (or channel centre) boundary condition at high Reynolds numbers. Other features, like the location of the peak of u rms, also matched well with reported literature.

We note that with the composite model (equation 12) the results for the inner mode are nearly identical with those reported in S and V2. In addition, it was possible to obtain (damped) outer modes, which could be compared with the data of Hussain and Reynolds (1972). This will be the main focus of the discussion here.

As far as we are aware, the only experimental results with which direct comparisons may be made are those of Hussain and Reynolds (1972) in a turbulent channel. Hussain and Reynolds (1972) introduced a 2-D disturbance in a fully developed turbulent channel using a vibrating ribbon at four different frequencies and extracted the resulting eigenfunctions and their amplitudes and phases using phase-locked averaging. However, all the disturbances considered there (the minimum frequency chosen was 25Hz and the maximum frequency was 100Hz) were found to be damped outer (since c_r was found to be close to 1.0) modes. The average velocity, V, used in the experiment was 5.88m/s and the channel half-width, δ was approximately 3.2cm yielding a Reynolds number of 12160.

Our computations show that in essence there exist two modes; an inner mode (with c_r close to 0.3) and an outer mode (with c_r close to 1.0) which is always damped (there exist additional outer modes which are more heavily damped which will not be of concern here). Figure 3 shows the growth rate, **for the inner mode**, as a function of the wave number (both non-dimensionalised in inner variables) computed at a Reynolds number of 5000. We see that there exists a large band of unstable wave numbers. The curve shown in figure 3 is nearly universal (c.f. S and V and S and V2) and thus we can use it to compare with the experiments of Reynolds and Hussain (1972) even though their flow Reynolds number is different. Given the parameters as in Hussain and Reynolds (1972), we estimate that a band of unstable waves (inner mode) exists for the frequency range, approximately 150Hz to approximately 1kHz, with the peak at 440Hz. Further, till the frequency becomes close to 150Hz the inner modes are more heavily damped than the outer mode because the growth rate curve crosses the abscissa rather steeply. Direct comparisons with the experimental data of Hussain and Reynolds (1972) are presented in Table 1.

We note that the temporal growth rate exponent in the computations is estimated as αc_i since a real α is chosen. However, in the experiment it is the frequency, αc, that is real, and α is a complex number. Thus the growth rate exponent in the experiment is - $\alpha_i x$. We

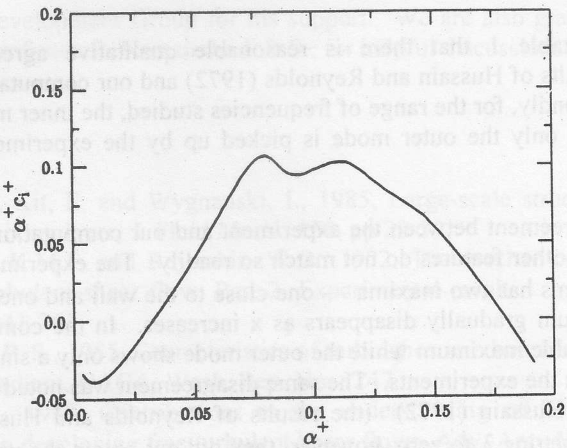

Figure 3: Graph of the growth rate, $\alpha^+ c_i^+$ versus α^+, (inner variables scaling)

can convert this to a temporal growth rate by writing $x = V_g t$, where, V_g is the group velocity, given by $V_g = d(c\alpha)/d(\alpha)$. However, since for both the inner and outer modes we find that c_r varies very slowly with α, the group velocity $V_g \approx c_r$, from which we can estimate the equivalent temporal growth rate exponent in the experiment as $-\alpha_i c_r$. This is the growth rate reported in Table .

Freq. (Hz)	α_r	c_r	Growth rate	Source
25	0.981 ± 13%	0.861 ± 13%	- 0.063 ± 23%	H&R
50	1.87 ± 8.5%	0.904 ± 8.5%	- 0.11 ± 20.5%	H&R
75	2.72 ± 7%	0.93 ± 7%	- 0.18 ± 21%	H&R
100	3.57 ± 6.1%	0.946 ± 6.1%	- 0.27 ± 22%	H&R
24.5	0.85	0.98	- 0.049	outer mode
50.7	1.68	1.02	- 0.041	outer mode
77.0	2.49	1.05	- 0.052	outer mode
99.6	3.18	1.06	- 0.085	outer mode
25.5	1.69	0.265	- 0.62	inner mode
49.6	3.60	0.242	- 0.60	inner mode
74.3	5.11	0.255	- 0.47	inner mode
98.1	6.45	0.267	- 0.33	inner mode

Table 1. Wave data: Comparison between experiments of Hussain and Reynolds (1972) (H and R) and present calculations. All quantities except the frequency are dimensionless and normalised using the channel half width and the mean velocity.

We note from table 1 that there is reasonable qualitative agreement between the experimental results of Hussain and Reynolds (1972) and our computational results for the outer mode. Secondly, for the range of frequencies studied, the inner mode is more heavily damped and thus only the outer mode is picked up by the experiments of Hussain and Reynolds (1972).

While there is agreement between the experiment and our computation for the parameters shown in table 1, other features do not match so readily. The experimental data show that the profile for u rms has two maxima — one close to the wall and one in the outer region. The inner maximum gradually disappears as x increases. In the computations, the inner mode shows a double maximum while the outer mode shows only a single peak which is in disagreement with the experiments. The same disagreement was noted in the computations of Reynolds and Hussain (1972) (the results of Reynolds and Hussain (1972) can be obtained here by setting λ to zero, however, they did not obtain the unstable inner mode). Reynolds and Hussain (1972) tried to resolve this by noting that there is a second outer mode that is more heavily damped and if initially the ribbon excites both the modes then one could obtain a double humped profile as seen in the experiments with the hump near the wall getting suppressed as x increases, because it is more heavily damped. However, as shown in Reynolds and Hussain (1972) even with the superposition of the outer modes, the locations of the two maxima do not agree with the experimentally observed values. It is likely that it is a superposition of the outer and inner modes determined here, that is more appropriate, especially at 100Hz, since the inner mode is only slightly more damped than the outer mode at this frequency. This can be properly resolved only after conducting further experiments at higher frequencies designed to match those of the unstable inner mode.

Finally, we note from the spectral measurements of Hussain and Reynolds (1975), that the band of unstable wavenumbers reported here (in Figure 3) is fully contained in the energy-containing eddies region (figure from Hussain and Reynolds (1972) not shown). The same result was reported in S and V for the turbulent boundary layer.

CONCLUSIONS

We have shown that the extended Orr-Sommerfeld equation of S and V is capable of capturing some of the key features of turbulent channel flows. The composite expression for the anisotropy function (λ) reported here is adequate in capturing the behaviour both close to the wall and away from it. Comparisons with the experimental data of Reynolds and Hussain (1972) \ show good qualitative agreement, however, further experiments at higher frequencies are needed to verify our computations for the inner mode.

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REFERENCES

[1] Gaster, M., Kit, E. and Wygnanski, I., 1985, Large-scale structures in a forced turbulent mixing layer, J. Fluid Mech., **150**, pp 23-39.

[2] Hussain, A.K.M.F. and Reynolds, W.C., 1972, The mechanics of an organised wave in turbulent shear flow. Part 2. Experimental results. J. Fluid Mech., **54**, part 2, pp 241-261.

[3] Klebenoff, P. S., 1955, Characteristics of turbulence in a boundary layer with zero pressure gradient, NACA, Wash. Rep. No. 1247.

[4] Liu, J.T.C., 1988, Contributions to the understanding of large-scale coherent structures in developing free turbulent shear flows, Adv. App. Mech., **26**, pp 183-309.

[5] Malkus, W.V.R., 1956, Outline of a theory of turbulent shear flow, J. Fluid Mech., **1**, pp 521.

[6] Pope, S.B., 1975, A more general effective viscosity hypothesis, J. Fluid Mech., **72**, part 2, pp 331-340.

[7] Reynolds, W.C. and Hussain, A.K.M.F., 1972, The mechanics of an organized disturbance in turbulent shear flow. Part 3. Theoretical models and comparisons with experiments, J. Fluid Mech, **54**, part 2, pp 263-288.

[8] Reynolds, W.C. and Tiederman, W.G.,} 1967, Stability of turbulent channel flow, with application to Malkus's theory, J. Fluid Mech, **27**, part 2, pp 253-272.

[9] Roshko, A., 1992, Instability and turbulence in shear flows, *'Theoretical and Applied Mechanics'*, Eds. S. R. Bodner, J. Singer, A. Solan and Z. Hussain, Publ. Elsevier Science.

[10] Sen, P.K., Venkateswarulu, D. and Maji, S. 1985, On the stability of pipe-Poiseuille flow to finite amplitude axi-symmetric and non-axi-symmetric disturbances, J. Fluid Mech. **158**, pp. 289-316.

[11] Sen, P.K., and Vashist, T.K., 1989, On the nonlinear stability of boundary-layer flow over a flat plate, Proc. Roy. Soc. Lond., A **424**, pp. 81-92.

[12] Sen, P.K. and Veeravalli, S.V., 1998a, On the behaviour of organised disturbances in a turbulent boundary layer, Sādhanā, Vol. **23**, Part 2, pp. 167-193.

[13] Sen, P.K. and Veeravalli, S.V., 1998b, Evolution of Organised disturbances in Fully Developed Turbulent Channel Flow, Proc. 25th National Conference on Fluid Mechanics and Fluid Power, I I T Delhi, pp 853 - 860.

[14] Spalart, P.R., 1988, Direct Numerical simulation of a turbulent boundary layer up to $R_\theta = 1410$, J. Fluid Mech., **187**, pp 61-98.