Derivation of Generalized Variational Principles of Elasticity Via Liu's Systematic Method

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Abstract: With Liu's systematic method for various generalized variational principles in fluid mechanics, we establish a more general form of functional of generalized variational principle of elasticity directly from its governing equations and boundary conditions. In particular, this new deduced functional can be reduced into known functionals of existing generalized variational principles in elasticity including Chien's generalized variational principle. The authors also point out the physical meanings of Liu's trial-functional. Liu's method can be converted into He's semi-inverse method, which, without using Lagrange multipliers, has proved to be as effective and convenient as Liu's.

Keywords: *Variational principles in elasticity, Semi-Inverse Method, Trial-Function*

INTRODUCTION

Generally speaking, there exist two basic ways to describe a physical problem: 1) by partial differential equations (PDEs) with boundary or initial conditions (BC or IC); 2) by variational principles (VPs). PDE model requires strong local differentiability (smoothness) of the physical field, while its VP partner requires weaker local smoothness or only local integrability. For discontinuous field, the PDE model is no longer valid, while its VP partner is powerfully applied. Moreover the VP model has many advantages over its PDE partner: simple and compact in form while comprehensive in content, encompassing implicitly almost all information characterizing the problem under consideration PDEs and natural BC/IC; capable of hinting naturally how the boundary/initial value problem should be properly posed. Applying variational principle with variable-domain, we can powerfully deal with discontinuities in complex materials. It is also a sound theoretical foundation of the finite element method (FEM) and other direct variational methods such as Ritz's, Trefftz's, and Kantorovitch's methods.

It is well known that, in general, it is extremely difficult to deduce a generalized variational principle directly from its governing equations and boundary conditions or initial conditions. Much attention has been put on the existence and uniqueness for the inverse problem of calculus of variations and ways to search for its variational principle of a physical problem. According to Vainberg's theorem, the VPs for a physical problem exist and can be constructed formally, if the differential operators in the PDE-formulation are symmetric. Such a requirement is overly restrictive, and it is important that we remove it if possible.

Hu[1], using the so-called trial-and-error method , obtained the well-known Hu-Washizu principle. The method is to try to search for a functional, then making the functional stationary to see whether its stationary conditions (Euler's equations) satisfy its governing equations and boundary conditions. A more scientific approach. Lagrange multiplier method, was discussed by Chien in Ref[2], with such method the constraints in a conditioned variational principle can be removed by the Lagrange multipliers. Washizu[3] first used such method and established Hu-Washizu principle.

However, for some physical problems, no known variational principle is at hand. The Lagrange multiplier method, therefore, loses its power in such case. Moreover, in using Lagrange multiplier method to arrive at a GVP, one may always come across variational crisis[4~7] (some of Lagrange multipliers become zero, and thus fail to reach its aim), which was found by Chien[4] in elasticity, and Liu[5] and He[6,7] in fluid mechanics. Various method have been proposed to eliminate the crisis, for example, high-order Lagrange multiplier method by Chien[4], preconditioned method by Liu[5], and semi-inverse method by He[6,7].

In 1990, Liu[8] proposed a more systematic method to establish various VPs in fluid mechanics, and great progress was made especially in turbomachinery and aerodynamics. Details can be found in Liu's publications. In this paper, the author illustrates Liu's method to establish GVPs of elasticity , and proposes a modified Liu's method to make it more effective and convenient.

AN INTRODUCTION TO LIU'S METHOD

Liu's method is illustrated using an incompressible rotational flow. The governing equations for this problem are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\Phi_x = u\,, \tag{2}$$

$$\Phi_y = v + f \tag{3}$$

where u,v denote velocity in x-and y-directions respectively, f is vertex and Φ is pseudo-potential function.

Here we want to search for a functional whose stationary conditions satisfy the above equations (1) , (2) and (3). According to Liu [8], we can assume that there exists a functional like (here the boundary conditions is not taken into consideration),

$$J(u,v,\Phi,\lambda,\alpha,\beta)=\iint\left\{F+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}\mathrm{d}x\,\mathrm{d}y$$

$$+\iint\left\{\alpha(\Phi_x-u)+\beta(\Phi_y-v-f)\right\}\mathrm{d}x\,\mathrm{d}y \tag{4}$$

where λ,α,β are Lagrange multipliers, F is an unknown to be determined.

We call above functional as trial-functional or energy trial-functional, whose stationary conditions should satisfy equations (1), (2) and (3). Making the above trial-functional stationary, we can obtain following stationary conditions:

$\delta\lambda$: equation (1);

$\delta\alpha$: equation (2);

$\delta\beta$: equation (3);

$$\delta u: \frac{\partial F}{\partial u} - \frac{\partial \lambda}{\partial x} - \alpha = 0 \tag{5}$$

$$\delta v: \frac{\partial F}{\partial v} - \frac{\partial \lambda}{\partial y} - \beta = 0 \tag{6}$$

$$\delta \Phi: \frac{\partial F}{\partial \Phi} - \frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y} = 0 \tag{7}$$

The Lagrange multipliers can be preliminarily determined as follows:

$$\alpha = u \quad , \quad \beta = v \quad , \quad \lambda = -\Phi \tag{8}$$

and the equations (5) to (7) reduce to :

$$\frac{\partial F}{\partial u} = 0 \tag{9}$$

$$\frac{\partial F}{\partial v} = -f \tag{10}$$

$$\frac{\partial F}{\partial \Phi} = 0 \tag{11}$$

Hence we can identify the unknown F:

$$F = -fv \tag{12}$$

The following VP , therefore, can be obtained:

$$J(u,v,\Phi) = \iint\left\{-\Phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u(\Phi_x - u)\right\} \mathrm{d}x\,\mathrm{d}y + \iint\left\{v(\Phi_y - v - f) - fv\right\} \mathrm{d}x\,\mathrm{d}y \tag{13}$$

It is easy to prove that its stationary conditions (Euler's equations) satisfy equations (1), (2) and (3).

MATHEMATICAL FORMULATION OF SMALL DISPLACEMENT PROBLEM IN ELASTICITY

Let τ be the volume of nonlinear elastic body subjected to the action of distributed body forces f_i (i=1,2,3) , Γ_σ be the portion of boundary surface subjected to the action of external forces $\overline{p}_i$, and Γ_u be the other portion of boundary surface where the displacements $\overline{u}_i$ are given. Under static equilibrium, the stresses σ_{ij}, strains e_{ij} and displacement u_i satisfy the following five sets of conditions, namely

1) Equilibrium conditions:

$$\sigma_{ij,j} + f_i = 0 \qquad (\text{in} \quad \tau) \tag{14}$$

in which $\sigma_{ij,j} = \partial\sigma_{ij} / \partial x_j$.

2) Stress-strain relations: For linear elasticity, we have

$$\sigma_{ij} = a_{ijkl} e_{kl} \qquad (\text{in} \quad \tau) \tag{15}$$

or

$$e_{ij} = b_{ijkl} \sigma_{kl} \qquad (\text{in} \quad \tau) \tag{16}$$

in which a_{ijkl}, b_{ijkl} represent elastic and compliance tensors respectively.

Let us now introduce the strain energy density A and complementary B. They are defined in general by

$$A = \int_0^e \sigma_{ij}\, d e_{ij} = \frac{1}{2} e_{ij} a_{ijkl} e_{kl} \quad \text{or} \quad \frac{\partial A}{\partial e_{ij}} = \sigma_{ij} \tag{17}$$

$$B = \int_0^\sigma e_{ij}\, d\sigma_{ij} = \frac{1}{2} \sigma_{ij} b_{ijkl} \sigma_{kl} \quad \text{or} \quad \frac{\partial B}{\partial \sigma_{ij}} = e_{ij} \tag{18}$$

and satisfy the following energy identity

$$A + B = e_{ij}\sigma_{ij} \tag{19}$$

3) Strain-displacement relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (\text{in} \quad \tau) \tag{20}$$

4) Boundary conditions for given surface displacement

$$u_i = \overline{u}_i \qquad (\text{in} \quad \Gamma_u) \tag{21}$$

5) Boundary conditions for given external force on boundary surface

$$\sigma_{ij} n_j = \overline{p}_i \qquad (\text{in} \quad \Gamma_\sigma) \tag{22}$$

DERIVATION OF GVP VIA LIU'S METHOD

In this section Liu's method is illustrated to establish a more general functional of generalized variational principle. According to Liu, a trial-functional can be constructed as follows:

$$J(\sigma_{ij}, e_{ij}, u_i, \alpha_{ij}, \beta_{ij}, \lambda_i) = \iiint \left\{ F + \alpha_{ij}(e_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}) \right\} d\tau + \iiint \left\{ \beta_{ij}(\sigma_{ij} - a_{ijmn} e_{mn}) \right\} d\tau$$

$$+\iiint\{\lambda_i(\sigma_{ij,j}+f_i)\}\,d\tau+I_B \tag{23}$$

where $\alpha_{ij},\beta_{ij}\ \lambda_i$ are the Lagrange multipliers and I_B is a boundary integral.

Making of the above trial-functional stationary to identify the Lagrange multipliers yields

$\delta\alpha_{ij}$: equation (20);

$\delta\beta_{ij}$:equation(15);

$\delta\lambda_i$: equations(14);

$$\delta\sigma_{ij}:\frac{\delta F}{\delta\sigma_{ij}}+\beta_{ij}-\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})=0 \tag{24}$$

$$\delta e_{ij}:\frac{\delta F}{\delta e_{ij}}+\alpha_{ij}-\beta_{mn}a_{mnij}=0 \tag{25}$$

$$\delta u_i:\frac{\delta F}{\delta u_i}+\alpha_{ij,j}=0 \tag{26}$$

Here,

$$\frac{\delta F}{\delta\xi}=\frac{\partial F}{\partial\xi}-\frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial\xi_{,j}}\right)=\frac{\partial F}{\partial\xi}-\left(\frac{\partial F}{\partial\xi_{,j}}\right)_{,j} \tag{27}$$

which reduces to partial differential, $\delta F/\delta\xi=\partial F/\partial\xi$ when F is expressed implicitly with $\xi_{,j}$. ξ denotes a kind of independent function.

From the form of the equations (24) to (26), we can identify preliminarily the multipliers as follows:

$$\alpha_{ij}=p\sigma_{ij},\beta_{ij}=qe_{ij},\ \lambda_i=ru_i \tag{28}$$

where p,q,r are constants, and $p+r\neq 0, p+q\neq 0$.

Consequently, the trial-functional (4.1) can be renewed as follows:

$$J(\sigma_{ij},e_{ij},u_i)=$$

$$\iiint\left\{F+p\sigma_{ij}(e_{ij}-\frac{1}{2}u_{i,j}-\frac{1}{2}u_{j,i})\right\}d\tau$$

$$+\iiint\{qe_{ij}(\sigma_{ij}-a_{ijmn}e_{mn})\}\,d\tau$$

$$+\iiint\{ru_i(\sigma_{ij,j}+f_i)\}\,d\tau+I_B \tag{29}$$

The above integral has the form of energy that is why the trial-functional is also called energy trial-functional.

The stationary conditions of the above renewed trial-functional should satisfy equations (14) to (20) in τ , accordingly, its trial-Euler's equations can reduced to

$$\frac{\delta F}{\delta \sigma_{ij}} = -(q-r)e_{ij} \tag{30}$$

$$\frac{\delta F}{\delta e_{ij}} = -(p-q)\sigma_{ij} \tag{31}$$

$$\frac{\delta F}{\delta u_i} = pf_i \tag{32}$$

From equations (4.8)~(4.9), we can identify the unknown F as follows:

$$F = -(q-r)B - (p-q)A + pf_i u_i \tag{33}$$

Consequently, after identifying the boundary integral, discussed below, we can obtain the following more general functional of generalized variational principle,

$$\begin{aligned} J(\sigma_{ij}, e_{ij}, u_i) = \\ -p \iiint \left\{ A - \sigma_{ij}(e_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}) - f_i u_i \right\} d\tau \\ -q \iiint (B + A - e_{ij}\sigma_{ij}) d\tau \\ +r \iiint \left\{ B + u_i(\sigma_{ij,j} + f_i) \right\} d\tau \\ +p \iint_{\Gamma_u} \sigma_{ij} n_j (u_i - \bar{u}_i) dS + p \iint_{\Gamma_\sigma} \bar{p}_i u_i dS \\ -r \iint_{\Gamma_u} \sigma_{ij} n_j \bar{u}_i dS + r \iint_{\Gamma_\sigma} (\bar{p}_i - \sigma_{ij} n_j) u_i dS \end{aligned} \tag{34}$$

Its stationary conditions give

$$\delta\sigma_{ij}: \ p(e_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}) - q(\frac{\partial B}{\partial \sigma_{ij}} - e_{ij})$$

$$+ r(\frac{\partial B}{\partial \sigma_{ij}} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}) = 0 \tag{35}$$

$$\delta e_{ij}: \ -(p+q)(\frac{\partial A}{\partial e_{ij}} - \sigma_{ij}) = 0 \tag{36}$$

$$\delta u_i: \ (p+r)(\sigma_{ij,j} + f_i) = 0 \tag{37}$$

$$\text{at } \Gamma_u: (p+r)(u_i - \bar{u}_i) = 0 \tag{38}$$

at $\Gamma_\sigma: (p+r)(\bar{p}_i - \sigma_{ij} n_j) = 0$ (39)

which satisfy equations (14) to (22) respectively.

In particular, this new functional (34) can be reduced into Chien's generalized variational principle $(r = 0, p = -1)$, Hu-Washizu principle $(r = 0, p = -1, q = 0)$ and Hellinger-Reissner principle $(r = 0, p = 0, q = 1)$.

MODIFIED LIU'S METHOD

In this section we explain the physical meaning of the unknown F in Liu's trial-functional. Supposing that there exists an unknown variational principle with only one kind of independent variation (the boundary conditions is not taken into consideration until at the end of this section), we have

$$J(\xi) = \iiint F(\sigma_{ij}, \sigma_{ij,j}, e_{ij}, e_{i,jj}, u_i, u_{i,j})\,d\tau \tag{40}$$

where ξ is an independent function, either stresses or strains or displacement, and F is an unknown to be determined.

Its stationary conditions correspond to one of three governing equations, say, equations (14), then the other equations, (15) to (20), in τ are the variational constraints. Use of Lagrange multipliers to remove the constraints yields following trial-functional:

$$J(\sigma_{ij}, e_{ij}, u_i, \alpha_{ij}, \beta_{ij}) = \iiint \left\{ F + \alpha_{ij}(e_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}) \right\} d\tau + \iiint \left\{ \beta_{ij}(\sigma_{ij} - a_{ijmn} e_{mn}) \right\} d\tau \tag{41}$$

where α_{ij}, β_{ij} are the Lagrange multipliers.

By the same manipulation, we preliminarily identify the multipliers as follows:

$$\alpha_{ij} = p\sigma_{ij}, \beta_{ij} = qe_{ij} \tag{42}$$

Thus the trial-functional (41) can be renewed as follows:

$$J(\sigma_{ij}, e_{ij}, u_i) = \iiint \left\{ F + p\sigma_{ij}(e_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}) \right\} d\tau + \iiint \left\{ qe_{ij}(\sigma_{ij} - a_{ijmn} e_{mn}) \right\} d\tau \tag{43}$$

A careful inspection and analysis of the trial-functionals (29) and (43) reveal that the energy integral can be readily constructed directly from its governing equations. This interesting discovery leads to a modified Liu's method or semi-inverse method, which has found its application not only in fluid mechanics[8~15] , but also in elasticity[16,17].

The identification of the unknown F is very similar to that of Lagrange multipliers. Making the above trial-functional stationary yields following Euler's equations:

$$p(e_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}) + qe_{ij} + \frac{\delta F}{\delta \sigma_{ij}} = 0 \tag{44}$$

$$p\sigma_{ij} + q(\sigma_{ij} - a_{ijmn}e_{mn}) - qe_{mn}a_{mnij} + \frac{\delta F}{\delta e_{ij}} = 0 \tag{45}$$

$$p\sigma_{ij,j} + \frac{\delta F}{\delta u_i} = 0 \tag{46}$$

The above equations with unknown F is called as trial-Euler's equations, which should satisfy the equation (14), (15) to (19) and (20). Accordingly, equations (51), (53) and (54) reduce to

$$\frac{\delta F}{\delta \sigma_{ij}} = -qe_{ij} \tag{47}$$

$$\frac{\delta F}{\delta e_{ij}} = (-p + q)\sigma_{ij} \tag{48}$$

$$\frac{\delta F}{\delta u_i} = p\bar{f}_i \tag{49}$$

Therefore, the unknown F can be readily identified as follows:

$$F = -qB + (q - p)A + p\bar{f}_i u_i \tag{50}$$

Following GVP, we then obtain,

$$\begin{aligned} J(\sigma_{ij}, e_{ij}, u_i) = & \\ & \iiint\left\{ -qB + (q-p)A + p\bar{f}_i u_i \right\} d\tau \\ & + \iiint\left\{ p\sigma_{ij}(e_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{i,j}) \right\} d\tau \\ + & \iiint\left\{ qe_{ij}(\sigma_{ij} - a_{ijmn}e_{mn}) \right\} d\tau \end{aligned} \tag{51}$$

As $A = e_{ij}a_{ijmn}e_{mn} / 2$, the above functional can be rewritten as follows:

$$\begin{aligned} & J(\sigma_{ij}, e_{ij}, u_i) = \\ & -p\iiint\left\{ A - \bar{f}_i u_i - \sigma_{ij}(e_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}) \right\} d\tau \end{aligned}$$

$$-q\iiint (B + A - e_{ij}\sigma_{ij})\,d\tau \tag{52}$$

This can be converted into Chien variational principle when $p = 1,\ q = \lambda$:

$$J_{Chien}(\sigma_{ij}, e_{ij}, u_i) =$$

$$\iiint\left\{-A + \bar{f}_i u_i + \sigma_{ij}(e_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i})\right\}d\tau$$

$$+\iiint\left\{\lambda\left(-B - A + e_{ij}\sigma_{ij}\right)\right\}d\tau \tag{53}$$

Now we illustrate how to eliminate the constraints of boundary conditions by semi-inverse method. The trial-functional can be constructed as follows:

$$J_{GVP}(\sigma_{ij}, e_{ij}, u_i) = J_{Chien}(\sigma_{ij}, e_{ij}, u_i)$$

$$+\iint_{\Gamma_u} G\,dS + \iint_{\Gamma_\sigma} H\,dS \tag{54}$$

where G and H are unknowns .

Making the above trail-functional (54) stationary, and using the Green's theory at the boundary, one can obtain following stationary conditions:

At the boundary Γ_u ,

$$\delta u_i: \quad -\sigma_{ij}n_j + \frac{\delta G}{\delta u_i} = 0 \tag{55}$$

$$\delta\sigma_{ij}: \quad \frac{\delta G}{\delta\sigma_{ij}} = 0 \tag{56}$$

which should satisfy the boundary conditions (21), that is,

$$\frac{\delta G}{\delta\sigma_{ij}} = 0 = u_i - \bar{u}_i \tag{57}$$

From (55) and (57), the unknown G can be determined as follows:

$$G = \sigma_{ij}n_j(u_i - \bar{u}_i) \tag{58}$$

By the same manipulation, at the boundary Γ_σ, we have

$$\delta u_i: \quad \frac{\delta H}{\delta u_i} = \sigma_{ij}n_j = \bar{p}_i \tag{59}$$

$$\delta\sigma_{ij}: \quad \frac{\delta H}{\delta\sigma_{ij}} = 0 \tag{60}$$

Accordingly, the unknown H can be identified as follows:

$$H = \bar{p}_i u_i \tag{61}$$

Therefore the following GVP with more general form of functional can be obtained:

$$J(\sigma_{ij}, e_{ij}, u_i) =$$

$$\iiint \left\{ -A + \bar{f}_i u_i + \sigma_{ij}(e_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}) \right\} \mathrm{d}\tau$$

$$+ \iiint \lambda(-B - A + e_{ij}\sigma_{ij}) \mathrm{d}\tau$$

$$+ \iint_{\Gamma_u} \sigma_{ij} n_j (u_i - \bar{u}_i) \mathrm{d}S + \iint_{\Gamma_\sigma} \bar{p}_i u_i \mathrm{d}S \tag{62}$$

By constraint-recovering method [8], substituting (19) into the above functional yields well-known Hu-Washizu principle under the constraints of equation (19):

$$J_{\mathrm{HW}}(e_{ij}, u_i) =$$

$$\iiint \left\{ -A + \bar{f}_i u_i + \sigma_{ij}(e_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}) \right\} \mathrm{d}\tau$$

$$+ \iint_{\Gamma_u} \sigma_{ij} n_j (u_i - \bar{u}_i) \, \mathrm{d}S + \iint_{\Gamma_\sigma} \bar{p}_i u_i \, \mathrm{d}S \tag{63}$$

Integrating by parts,

$$J(e_{ij}, u_i) = \iiint \left\{ -A + \bar{f}_i u_i + \sigma_{ij} e_{ij} + \sigma_{ij,j} u_i \right\} \mathrm{d}\tau$$

$$+ \iint_{\Gamma_u} \sigma_{ij} n_j (u_i - \bar{u}_i) \, \mathrm{d}S + \iint_{\Gamma_\sigma} \bar{p}_i u_i \, \mathrm{d}S$$

$$- \oiint_{\partial\tau} \sigma_{ij} n_j u_i \, \mathrm{d}S \tag{64}$$

or

$$J(e_{ij}, u_i) = \iiint \left\{ -A + \bar{f}_i u_i + \sigma_{ij} e_{ij} + \sigma_{ij,j} u_i \right\} \mathrm{d}\tau$$

$$- \iint_{\Gamma_u} \sigma_{ij} n_j \bar{u}_i \, \mathrm{d}S + \iint_{\Gamma_\sigma} (\bar{p}_i - \sigma_{ij} n_j) u_i \, \mathrm{d}S \tag{65}$$

Substitution of constraint equations (19) into the above functional, without changing its stationary conditions, yields the well-known Hellinger-Reissner principle,

$$J_{\mathrm{HR}}(\sigma_{ij}, u_i) = \iiint \left\{ B + (\bar{f}_i + \sigma_{ij,j}) u_i \right\} \mathrm{d}\tau - \iint_{\Gamma_u} \sigma_{ij} n_j \bar{u}_i \, \mathrm{d}S + \iint_{\Gamma_\sigma} (\bar{p}_i - \sigma_{ij} n_j) u_i \, \mathrm{d}S \tag{66}$$

So, the Hu-Washizu principle and Hellinger-Reissner principle are all conditioned variational principles under the constraints of stress-strain relations [4,17].

CONCLUSION

In this paper, the author has successfully applied Liu's systematic method to elasticity. As a result, a more genera functional of generalized variational principle, which was unknown to us till now, has been obtained. The results can be extended readily to nonlinear elasticity.

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