Some Thoughts on the Linear and Non-Linear Stability of Parallel and Near-Parallel Flows

P. K. Sen and **Shriram Hegde**

Department of Applied Mechanics
Indian Institute of Technology, Delhi
New Delhi 110-016, INDIA

Abstract: This paper discusses, the Orr-Sommerfeld (OS) equation, extended forms of the OS equation due to non-parallel and non-linear effects, and discusses the relevance of the Gaster transform for non-parallel and non-linear cases. It is seen that, within `reasonable' limits, the Gaster transform for equivalence of spatial and temporal problems continues to hold for both non-parallel and non-linear cases. Also, a simplified procedure is outlined for dealing with non-linear problems.

Keywords: *Hydrodynamic stability; Linear, Non-linear, Non parallel effects, Gaster's theorem*

INTRODUCTION

Over the last several decades much has been contributed to the understanding of the phenomenon of laminar to turbulent transition, in parallel and near-parallel laminar shear flows, by the use of the Orr-Sommerfeld (OS) equation. Here we discuss some aspects of stability calculations that may be made, using the OS equation and extended versions of the OS equation accounting for non-parallel and non-linear effects. Our attention will be confined to wall-bounded flows, like channel flow and boundary layer flow, although, the results could carry over almost entirely to cases of free shear flows as well.

One of the interesting theorems to look at, is the Gaster [2] transform citing the equivalence of the temporal and spatial problems. Within certain limits, this equivalence is seen to carry over to non-parallel and non-linear cases also.

We begin by giving a very brief introduction to the OS equation. Starting from the linearised Navier-Stokes and continuity equations, and introducing a 2D disturbance streamfunction ψ of the form $\psi \sim \phi(y)\exp[i(\alpha x - \beta t)]$, or $\psi \sim \phi(y)\exp[i\alpha(x - ct)]$, where α is the spatial wavenumber, β is the temporal frequency, $c = \beta / \alpha$ is the phase speed, and *x, y* are co-ordinates respectively along the flow and normal to the wall, one obtains the OS equation given below:

$$i\alpha\left[(u - c)\left(\phi'' - \alpha^2\phi\right) - u''\phi\right] - \frac{1}{R}\left(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi\right) = 0 \; ; \qquad (1)$$

where, in (1), primes (') denote differentiation with respect to y, and R is the Reynolds number based on a suitable length scale and suitable velocity scale pertinent to a particular problem. Two dimensional disturbances are considered in view of the well known Squire's theorem that, for linear stability theory, 2D disturbances are more unstable than 3D ones. Actually in the initial phase of the growth of linear disturbances, the disturbances may grow as 2D disturbances for quite some time, till C-type and H-type 3D bifurcations take over. We will confine our attention to 2D disturbances here.

The OS equation as given by (1) is exact only for a strictly parallel mean flow, like channel flow. For slightly non-parallel flows, like in boundary layer flows, the so-called 'quasi-parallel' approximation needs to be made, to get the form of (1). In this approximation the mean velocity v in the transverse direction is small and is ignored, i.e. $v \sim 0$. Also ignored are the weak changes with respect to x of the longitudinal mean flow velocity u, i.e. $\partial u / \partial x \sim 0$. However, at each local station the local distribution of u is used in the OS equation (1). This takes care of the so-called non-parallel effects to a large extent.

Before going into various aspects of the calculations based on the OS equation, and extended forms of the OS equations, it will be pertinent to look at an operational principle reported by Sen and Thomas [10]. This is discussed next.

AN OPERATIONAL PRINCIPLE

The OS equation (1), has to be solved with appropriate boundary conditions. At the wall the conditions are $\phi,\ \phi' = 0$. For channel flow, the centreline conditions are given as $\phi',\ \phi''' = 0$ for the (least stable) symmetric modes. For boundary layer flows the outer condition at the edge of the boundary layer is given by $\phi, \sim \exp(-\alpha y)$. The OS equation (1), along with its appropriate and relevant boundary conditions for a given problem, constitutes an eigenvalue problem which is equivalent to the existence of a functional relationship of the form $F(\alpha, \beta, R) = 0$. In other words one may make an initial choice of *any two* of the three parameters α, β, R, and, the third comes out as part of the answer for the solution of the eigenvalue problem. Further, for the general eigenvalue problem, all three of the variables α, β, R could be complex. For physically relevant problems however, the Reynolds number R is real as an automatic initial choice, given by the length and velocity scales of the problem. Thereafter, the functional relationship ensuing from the eigenvale problem reduces to either $\beta = \beta(\alpha)$, or, $\alpha = \alpha(\beta)$. Classically, this has given rise to two classes of problems, viz. the temporal problem and the spatial problem. In the temporal problem an initial choice of a real α is made, and β is obtained as an eigenvalue. In general β emerges as complex, i.e. $\beta = \beta_r + \beta_i$. As may be seen from the form $\psi \sim \phi(y) \exp[i(\alpha x - \beta t)]$, stability or instability is obtained according as $\beta_i < 0$ or $\beta_i > 0$. The temporal problem eigenvalue is also commonly obtained in terms of the phase velocity $c = c_r + c_i$. Since α is real, the sign of c_l determines stability or instability exactly as β_l does. Analogously, in the spatial problem an initial choice of real β is made, and α emerges, in general complex, as part of the eigenvalue problem, viz. $\alpha = \alpha_r + \alpha_i$. Again, stability or instability is obtained according as $\alpha_i > 0$ or $\alpha_i < 0$. Gaster [2] found an equivalence between the temporal and spatial problems, which aspect will be discussed later. Moreover the magnitude of α_l, or β_i, is small compared to the real parts, and *usually either of* α_i, β_i, c_l *is chosen as an important measure of a small parameter in the problem.* We will use this measure throughout in this paper.

Now, as more effects like non-parallelities and non-linearities are taken into account, the OS equation gets modified. The modifications are usually of $O(c_i)$. Let the original OS equation (1) be written in operator form as $L(\alpha, c)\phi = 0$, and let this equation be modified by an $O(c_i)$ term, viz. $G\phi$, where G is another operator, so that

$$L(\alpha, c)\phi = G\phi \tag{2}$$

New eigenvalues α, c may be found by solving the modified equation $(L - G)\ \phi = 0$. However, if $\hat{\alpha}, \hat{c}$ be the eigenvalues corresponding to the original OS equation, viz. $L(\hat{\alpha}, \hat{c})\phi = 0$, then, *our intention is to find the correction in* $\hat{\alpha}$ *or* $\hat{c}$, *that is brought about by the presence of the additional term* $G\phi$ *in (2)*. So we consider a variational form of (2), denote the corrections as c' and α' respectively for the temporal and spatial problems, and obtain the variational equations as follows:

$$L(\hat{\alpha}, \hat{c})\phi + c'\left[\left(\frac{\partial(L\phi)}{\partial c}\right)_{\alpha}\right]_{\hat{\alpha}, \hat{c}} = G\phi \ ; \tag{3}$$

$$L(\hat{\alpha}, \hat{c})\phi + \alpha'\left[\left(\frac{\partial(L\phi)}{\partial \alpha}\right)_{\beta}\right]_{\hat{\alpha}, \hat{c}} = G\phi \ ; \tag{4}$$

It is easy to work out the terms corresponding to c' and α', respectively in (3) and (4). These are given respectively as

$$L(\hat{\alpha}, \hat{c})\phi = c' i\alpha L_1(\hat{\alpha}, \hat{c})\phi + G\phi \ ; \tag{5}$$

where, $L_1\phi = (\phi'' - \alpha^2\phi)$; and,

$$L(\hat{\alpha}, \hat{c})\phi = \alpha' i L_2(\hat{\alpha}, \hat{c})\phi + G\phi \ ; \tag{6}$$

where, ignoring O(1/R) terms, $L_2\phi = -\left[u\left(D^2 - \alpha^2\right) - 2\alpha^2(u - c) - u''\right]\phi$ where, $D = d/dy$. The operators L_1 and L_2 are written in the above particular forms, to be compatible with earlier published literature.

The above equations (5) and (6) lead to a very simple way of determining either of the corrections c' or α'. This is based on the *solvability condition* of the equation $L(\hat{\alpha}, \hat{c})\phi$ = RHS, when a non-zero right hand side exists. For the temporal problem this is given as:

$$\int_0^{up} \theta[c' i\alpha L_1(\hat{\alpha}, \hat{c})\phi + G\phi]\, dy = 0 \ ; \tag{7}$$

and, for the spatial problem this is given as:

$$\int_0^{up} \theta[\alpha' i L_2(\hat{\alpha}, \hat{c})\phi + G\phi]\, dy = 0 \ ; \tag{8}$$

where, in (7) and (8), θ is the adjoint eigenfunction to the equation $L\phi = 0$. Also the upper limit of integration *up* is ∞ for the boundary layer case, and is the channel centreline for the channel flow case.

The results as given by (7) and (8) constitute a very powerful method of looking into various aspects of the stability problem, and this result is called herein the *Principle of Additive Augmentation.* That is to say that, if $G\phi$ is small ($O(c_i)$), then an additive correction results in the eigenvalue as given by either of equations (7) or (8). Further, the associated modification of the eigenfunction ϕ in view of the extra term $G\phi$ is small. We will look at various aspects of the stability problem based on this principle, in the rest of the paper.

GASTER'S THEOREM

Gaster's [2] theorem can be proved using the above principle. Let us conceive of a general spatio-temporal problem, which is equivalent to having `corrections' in both $\hat{c}$ and $\hat{\alpha}$. The resulting equation then becomes:

$$L(\hat{\alpha}, \hat{\beta})\phi - \beta' i L_1(\hat{\alpha}, \hat{\beta})\phi - \alpha' i L_2(\hat{\alpha}, \hat{\beta})\phi = 0 \quad ; \tag{9}$$

where, $\beta' = \alpha c'$, and notionally β' is the correction to $\hat{\beta}$, where as we know $\hat{\beta} = \hat{\alpha}\hat{c}$.

Now, let us imagine that we are looking at the temporal problem so that $\hat{\alpha}$ is real and $\hat{\beta}$ is complex. Our aim will be to derive the eigenvalue for the corresponding *spatial problem* based on the eigenvalue of the temporal problem. We now set $\beta' = -\beta_i$, because this renders β real. Thereafter, we try to find the `correction' in $\alpha'$, by treating the $\beta'$ term as though it were like the $G\phi$ term as in (6). The `correction' now comes from the solvability condition as in (8), and this is given as follows:

$$\alpha' = -\beta' \frac{\int_0^{up} \theta\left[L_1(\hat{\alpha}, \hat{\beta})\phi\right] dy}{\int_0^{up} \theta\left[L_2(\hat{\alpha}, \hat{\beta})\phi\right] dy}. \tag{10}$$

It is seen from (10) that the correction α', which can now be called α_i, has sign opposite to β_i. This is consistent with the respective signs of β_i and α_i being opposite, for stability or instability. The correction as in (10), could also have been obtained starting from the spatial problem and transforming to the temporal problem, and, exactly the same answer as in (10) would be obtained. Another observation that can be made is that the ratio of the two integrals in (10), is virtually entirely a real quantity, irrespective of the manner in which ϕ is normalised.

We will now show that the ratio of the two integrals is the same as the group velocity c_g, which is defined as $c_g = d\beta/d\alpha$. This will follow from a simplified `proof' of Gaster's theorem given below. Recalling that, at a given R, the eigenvalue problem gives $\beta = \beta(\alpha)$, it follows that

$$\delta\beta = \left(\frac{\mathrm{d}\beta}{\mathrm{d}\alpha}\right)\delta\alpha = c_g\delta\alpha \,. \tag{11}$$

Supposing now we start from the temporal problem in which β is complex, i.e. $\beta = \beta_r + \beta_i$. Next we choose $\delta\beta$ in (11) as $\delta\beta = -\beta_i$ is small in magnitude). This renders β_i real. And, the associated correction $\delta\alpha$ now becomes $\delta\alpha = \alpha_i$.

This completes the proof, of conversion of the temporal problem to the spatial problem, and the result is given by the well-known Gaster theorem [2]

$$-\beta_i = c_g\alpha_i \tag{12}$$

One important result that may be seen now directly is that the ratio of the two integrals in (10) is actually equal to the group velocity, that is

$$c_g = \frac{\int_0^{up} \theta\left[L_2\left(\hat{\alpha}, \hat{\beta}\right)\phi\right]dy}{\int_0^{up} \theta\left[L_1\left(\hat{\alpha}, \hat{\beta}\right)\phi\right]dy} \,. \tag{13}$$

This important result will be used at many places later on in this paper.

Another observation that may be made from (9-13) is that the result of converting from temporal to spatial problem, and *vice versa*, can be generalised to *any spatio-temporal case lying in between these two problems*. For instance if the correction β' in (10) is chosen as $\beta' = -\lambda\beta_i$, where λ is an arbitrary factor ~ O(1), then, the correction α' gives the eigenvalue of the spatio-temporal problem with $\beta = \hat{\beta} - \lambda\beta_i$ and $\alpha = \hat{\alpha} + \alpha'$.

We desist from dwelling on the question as to whether, in common physical situations, the spatial problem exists, or the temporal problem exists, or a spatio-temporal problem exists. Guided by experimental results, one may say that, for locally induced disturbances at a point, using for instance a vibrating ribbon, the spatial problem exists. However, what will be shown herein is that the *temporal problem can lead us to virtually all the answers corresponding to the spatial problem*, and the temporal problem is somewhat easier to solve.

THE NON-PARALLEL PROBLEM

Non-parallel effects come in when the mean flow is slightly diverging as in case of boundary layer flow over a flat plate. The problem may be formulated either by a fixed length scale (FLS) formulation, or by using a variable length scale (VLS) formulation, like using for instance the similarity variable. The final answers are no different using either formulation. Some typical results for non-parallel effects are reproduced herein (figs. 1-4) from Sen and Thomas [10] based on the adjoint method (A), and Sen et *al.* [11] based on energy methods (E). In these figures the experimental results of Klingmann *et al.* [6] are also shown. Also, in these figures, $F = (\beta/R) \times 10^6$ is the well-known frequency parameter. Our focus of interest will be the adjoint method, which is the same as the method based on the *Principle of Additive Augmentation* discussed herein.

One of the features of non-parallel analysis, as pointed out by Gaster [3] first, is that different monitorable properties (like the inner maximum of the disturbance velocity u_i, or the outer maximum of the disturbance velocity u_0) have slightly different growth rates. Neutral curves based on u_I and u_0 are shown in figs. (1-2). These subtle changes come about because, as one moves downstream along the plate, the local Reynolds number changes. Consequently, the eigenfunction continuously changes in shape. Thus, different monitorable properties show slightly different growth rates, and thus a set of neutral curves are obtained, rather than one single neutral curve.

Before discussing the above points further, let us look at the full non-parallel equation for ϕ, for the flat plate case, based say, on the fixed length scale (FLS) formulation. Here, δ_0 is a fixed length scale which is of the same numerical order as the boundary layer thickness, and δ is the (varying) boundary layer thickness given as $\delta \sim \sqrt{\nu x/U}$. One needs to make further the *Parabolized Stability Equation (PSE)* approximation (given by Bertolotti *et al.* [1]), which states that derivatives in *x* of ϕ, need be retained only upto first order, i.e. only $(\partial\phi / \partial x)$- terms need to be retained. Further, after retaining terms upto $O(R^{-1})$, the equation for ϕ is finally given as

$$L_{OS}\phi + L_{NP}\phi = L_2\left(\frac{\partial\phi}{\partial x}\right) ; \tag{14}$$

where L_{OS} is the Orr-sommerfeld operator referred to as L earlier, and L_{NP} is the additional non-parallel operator given as follows:

$$L_{NP}\phi = -\frac{1}{2R}\left[(u' + yu'')\phi' + (\bar{\phi} - yu)(\phi''' - \alpha^2\phi')\right] ; \tag{15}$$

where $\bar{\phi}$ is the Blasius stream-function for the mean flow. Experience shows that, if one settles for a little loss of accuracy, the non-parallel problem can still be defined quite faithfully even after dropping the $L_{NP}\phi$ term. The reduced problem is now given as

$$L(\alpha, c)\phi = L_2\left(\frac{\partial\phi}{\partial x}\right) ; \tag{16}$$

where the subscript 'OS' has been dropped from the operator $L(\alpha, c)$ in (16). We note therefore that even this reduced problem is given by a partial differential equation, and the question is what is one to do with the $(\partial\phi/\partial x)$ term. One way is to do marching in the *x* direction as outlined by Bertolotti et *al*. [1]. However, there is an alternative way of solving the problem as has been described in Sen and Thomas [10]. Some of the relevant parts of the Sen and Thomas method are discussed here.

First of all, in a problem like channel flow where the local Reynolds number remains unchanged in the downstream direction, there is no change in the shape of ϕ in the downstream direction. Therefore, one may set $(\partial\phi/\partial x) = 0$ in this particular problem, and, (16) reduces to the ordinary differential equation $L\phi = 0$. In the case with non-parallel effects, as for instance in boundary layer flow, the local Reynolds number changes downstream. Therefore, the *shape* of ϕ changes, though marginally, continuously downstream. We may treat this problem based on the principle of additive augmentation. Based on the spatial problem we have from (6) that

$$L(\bar{\alpha}, \bar{c})\phi = \alpha' i L_2(\bar{\alpha}, \bar{c})\phi + L_2\left(\frac{\partial\phi}{\partial x}\right). \tag{17}$$

The behaviour of ϕ may be given as

$$\frac{\partial\phi}{\partial x} = \lambda\phi + \chi \quad ; \tag{18}$$

where $L_2\chi$ is the shape change part and is by definition orthogonal to the eigensolution, viz.

$$\int_0^\infty \theta L_2\chi dy = 0 \,. \tag{19}$$

Substituting (18) in (17) one obtains

$$L(\bar{\alpha}, \bar{c})\phi = \alpha' i L_2(\bar{\alpha}, \bar{c})\phi + \lambda L_2\phi + L_2\chi \,. \tag{20}$$

The solvability condition of (20) is given as

$$\int_0^\infty \theta\left[i\alpha' L_2\phi + \lambda L_2\phi + L_2\chi\right] dy = 0 \,. \tag{21}$$

Note in (19) and (21) the upper limit of integration is ∞, i.e. for boundary layer problems. In view of (19), equation (21) gives the following important result, that,

$$\alpha' = i\lambda \,. \tag{22}$$

The result can be called the *Principle of Exchange of Growth Rates*. It stipulates that if a *size change* is proposed in ϕ, by a (exponential) λ, then, there is a corresponding *reduction* in the growth rate in the eigenvalue, by the same *factor* λ, in view of the correction α' being given by $\alpha' = i\lambda$.

The above thus leads us onto the concept of Optimal *Normalisation*, viz. keeping λ = 0. Which means that from station to station in *x*, the normalisation of the ϕ function should be so chosen that there is no *size change* in ϕ, and $(\partial\phi/\partial x) = \chi$ only. This condition is realised in numerical work by keeping

$$\int_0^\infty \theta L_2 \left(\frac{\partial \phi}{\partial x}\right) d = 0 \qquad (23)$$

Furthermore, after optimal normalisation has been carried out, there is no need to refine ϕ any further beyond what is given from the eigensolution corresponding to $L\phi = 0$. This is because, as has been mentioned in section 2 earlier, the correction in ϕ is marginal, since the RHS in (23) is small after optimal normalisation.

The above discussion leaves us in a happy position, because, after *optimal normalisation the spatial problem also reduces to the solution of a quasi-ordinary differential equation, namely* $L(\alpha, \beta) = 0$, *where* β *is real.*

Thus, once *both* the spatial and temporal problems are reduced to the solution of the equation $L(\alpha, \beta) = 0$, then, Gaster's theorem can be applied meaningfully to both, and, the spatial and temporal problem eigenvalues may be meaningfully interchanged as before, using the group velocity.

THE NON-LINEAR PROBLEM

We next consider the non-linear problem. In view of the discussions in section 3 above, we will make two simplifications at the outset. First, we will make the quasi-parallel assumption; and second, we will assume that optimal normalisation is being adopted so that $L_2(\partial\phi/\partial x)$ term may be neglected. A non-linear formulation may be obtained by expressing the disturbance ψ as a sum of the fundamental ϕ_1 and its harmonics as follows:

$$\psi = \sum_{n=-\infty}^{\infty} \phi_n \exp\{ni\alpha(x - ct)\} ; \qquad (24)$$

where, the fundamental is ϕ_1 and is given by the eigensolution of $L\phi_1 = 0$, ϕ_0 is the `zero-th' harmonic and represents the distortion in the mean motion, and, $\phi_{-n} = \widetilde{\phi}_n$, where (~) represents the complex conjugate. Upon substituting this form in the *full* two dimensional Navier-Stokes equations (not the linearized one), one obtains the following equation at any harmonic level `p', with $p \neq 0$, as follows:

$$ip\alpha\left[(u - c)\left(\phi_p'' - p^2\alpha^2\phi_p\right) - u''\phi_p\right] - \frac{1}{R}\left(\phi_p'''' - 2p^2\alpha^2\phi_p'' + p^4\alpha^4\phi_p\right)$$
$$= ip\alpha\left(NL_p\right) ; \qquad (25)$$

where the non-linear terms N_p are given as follows:

$$L_{p\alpha}\phi_p = ip\alpha(N_p) = ip\alpha\sum_{n=0}^{p}\left(-\frac{n}{p}\phi'_{(p-n)}f_n + \frac{p-n}{p}\phi_{(p-n)}f'_n\right)$$
$$+ip\alpha\sum_{n=1}^{\infty}\left(\frac{n}{p}\phi'_{(n+p)}\tilde{f}_n + \frac{n+p}{p}\phi_{(n+p)}\tilde{f}'_n\right)$$
$$+ip\alpha\sum_{n=(p+1)}^{\infty}\left(-\frac{n}{p}\tilde{\phi}'_{(n-p)}f_n - \frac{n-p}{p}\tilde{\phi}_{(n-p)}f'_n\right), \qquad (26)$$

where, $L_{p\alpha}$ is the left hand side operator in (25), $f_n = \phi''_n - n^2\alpha^2\phi_n$, and $p \geq 1$; $n \geq 0$.

It is customary to assume the ϕ_p to be slowly varying functions of either the time variable t or the space variable x. We ignore these effects in the harmonics $p \geq 2$, as these are equations forced by the fundamental and are forced solutions (i.e. non eigen-solutions) of Orr-Sommerfeld equations. However, for the fundamental equation for ϕ_1, the effect of slow variation in space or time is lumped together as either a correction α' in space, or c' in time, or in both. Therefore, similarly as in (9), the equation for $\phi_1$may be written as

$$L_\alpha(\hat{\alpha},\hat{c})\phi_1 - c' i\alpha L_1\phi_1 - \alpha' i L_2\phi_1 = i\alpha N_1, \qquad (27)$$

where, the operators L_1 and L_2 have been defined following equations (5) and (6). Further the operator L_α is the Orr-Sommerfeld operator, and $\hat{\alpha}$, $\hat{c}$ are eigenvalues corresponding to the eigensolution of the linear problem $L_\alpha(\alpha, c)\,\phi_1 = 0$. Also, the lumped non-linear term N_1, comprises harmonics *which are all forced by the fundamental*. Thus, the amplitude norm for any higher harmonic ϕ_p, $p \geq 2$ is given as $O(\varepsilon^p)$, where ε is the amplitude norm for the fundamental ϕ_1. Thus N_1 may be treated as being like a $G\phi$ term as in (2), since this term is intrinsically related to the amplitude of the fundamental term.

It needs to be remembered that c' and α' are both zero for the equations for the higher harmonics corresponding to $p \geq 2$, and the equation is given by

$$L_{p\alpha}(\hat{\alpha},\hat{c})\phi_p = ip\alpha N_p\,; \qquad p \geq 2\,; \qquad (28)$$

In (28) the operator $L_{p\alpha}$ contains the eigenvalues $\hat{\alpha}$, $\hat{c}$ corresponding to the fundamental ϕ_1, i.e. for $p = 1$. These are therefore not the eigenvalues for equations corresponding to $p \geq 2$. Thus there is no difficulty in obtaining the forced solutions for ϕ_p, from (28).

A special mention needs to be made regarding the zeroth harmonic, i.e the ϕ_0 term. We reproduce the general form of this equation, retaining the effect of slow variation with respect to t and x, as follows:

$$\frac{\partial \phi_0''}{\partial t} + u\frac{\partial \phi_0''}{\partial x} - u\frac{\partial \phi_0}{\partial x} - \frac{1}{R}\phi_0'''' = i\alpha\frac{\partial^2}{\partial y^2}\sum_{p=1}^{\infty} p\left(\phi_p\tilde{\phi}_p' - \tilde{\phi}_p\phi_p'\right). \tag{29}$$

The above equation has created some problems in the past, because, at least for the temporal problem the mean motion equation could give rise to singularities, as has been discussed in Sen and Venkateswarlu [8]. To illustrate, if it is assumed that the temporal problem has a a growth rate c_i given by the eigenvalue problem, and if the mean distortion in *u* is given as $u_d = \phi_0'$ then, by integrating (29) twice with respect to *y*, and assuming unchanged mean pressure gradient, one obtains the equation for u_d as follows:

$$2\alpha c_i u_d - \frac{1}{R}u_d'' = i\alpha\sum_{p=1}^{\infty} p\left(\phi_p\tilde{\phi}_p' - \tilde{\phi}_p\phi_p'\right). \tag{30}$$

The complementary equation

$$2\alpha c_i u_d - \frac{1}{R}u_d'' = 0 \;; \tag{31}$$

can have eigensolutions in channel flow, when $c_i < 0$, with boundary conditions $u_d = 0$; $y = 0$ and $u_d' = 0$; $y = 1$ (centreline). Further, similar eigensolutions can be obtained in boundary layer flow as well, as shown by Sen and Vashist [12]. When the complementary solution (31) exists, then, with (in general) a non-orthogonal right hand side as in (30) the solution blows up.

Actually the mean motion singularity problem can be avoided if the temporal problem is considered as a particular case from a general class of spatio-temporal problems as given by (29). From this perspective the boundary condition to be satisfied at the wall is that *both* $\phi_0 = 0$, $\phi_0' = 0$, $y = 0$. When this is stipulated the complementary equations

$$\frac{\partial \phi_0''}{\partial t} - \frac{1}{R}\phi_0'''' = 0 \;; \tag{32}$$

$$u\frac{\partial \phi_0''}{\partial x} - u''\frac{\partial \phi_0}{\partial x} - \frac{1}{R}\phi_0'''' = 0 \;; \tag{33}$$

respectively for the temporal and spatial problems, do not have eigensolutions.

Once the singularity problem for the mean motion is sorted out, one may consider neglecting the $\partial/\partial t$ and $\partial/\partial x$ terms in (29). One way to do this is by the equilibrium amplitude assumption of Reynolds and Potter [7], in which a finite equilibrium state is assumed ab *initio*. This is equivalent to stipulating ab *initio* in (27) that, $c_i + c' = 0$ in the temporal problem, or that, $\alpha_i + \alpha' = 0$ in the spatial problem. Even without making these assumptions the $\partial/\partial t$ and $\partial/\partial x$ terms in (29) are both $\sim O(c_i)$. This may also be seen in a plot of the modification in c_i, with the amplitude $|A|^2$, for instance in fig. (4) of Sen and Venkateswarlu.

Thus the simplified version of the mean motion equation (29) is given as

$$-\frac{1}{R}\phi_0''' = i\alpha \frac{\partial^2}{\partial y^2}\sum_{p=1}^{\infty} p\left(\phi_p\tilde{\phi}_p' - \tilde{\phi}_p\phi_p'\right) = N_0 . \qquad (34)$$

Looking at the equations (27), (28) and (34) one can now see, that, within limits of "reasonable" simplification, the non-linear equations for both the temporal and spatial problems are more or less of the *same* form. Some thought needs to go into not only the methods of solution of the non-linear problem, but also, how 'most' of the answer can be extracted based on simplified procedures.

There are two broad methods (apart from multi-deck asymptotic procedures) of solving the non-linear problem. One is by the amplitude expansion method postulated by Stuart [14] and Watson [15], based on which extensive calculations have been done by Herbert [5], Sen and Venakateswarlu [8], Sen *et al.* [9], Sen and Vashist [12], and others. In this method the ϕ_p functions are expanded in powers of the amplitude A of the fundamental. The growth rate is also expanded in powers of *A* by the Stuart-Landau equation, which is given for the temporal problem as follows:

$$\frac{dA}{dt} = \alpha c_i A + i\alpha A \sum_{n=1}^{\infty} K_n |A|^{2n}, \qquad (35)$$

where, K_n are called the Landau coefficients. An analogous expression can be sought for the spatial problem. If the simplified equations (27), (28) and (34) are considered, then, all *the Landau coefficients* K_n *are transformed from spatial to temporal problem, or vice versa, by means of the Gaster transform*. Thus, it is seen that *the Gaster tarnsform is valid for the non-linear problem based on the method of amplitude expansions.*

The amplitude expansion method, though originally intended to be a proper asymptotic expansion in terms of the amplitude, is not a fully rational method and eventually depends on the numerical convergence of the Stuart - Landau equation. Details of numerical work may be seen in the papers of Sen and others listed above. Perhaps a more direct method has evolved from the works of Herbert [4], culminating in the work of Bertolotti *et al.* [1]. In this method the starting point is usually equations like (27), (28) and (34), with non-parallel effects explicitly included. The equations are solved by marching along x. The amplitude at the next station is obtained from the growth rate estimated at the previous station.

The method of Bertolotti *et al.* [1] can be simplified even further by making minimal compromises. First, optimal normalisation need be used for ϕ_1 at each station in x. Second, for the spatial problem, the correction α' can be found from the solvability condition of (27), namely

$$\int_0^{\infty} \theta\left(\alpha' L_2\phi_1 + \alpha N_1\right) dy = 0 . \qquad (36)$$

Thirdly, accepting from the earlier discussions on the *Principle of Additive Augmentation* that, the shape change in the eigenfunction is relatively small, no further iteration is needed for ϕ_1 beyond what is given by the linear eigensolution. With these compromises, an overall non-linear growth rate can be estimated at each station by using the correction α' at each station. Finally it is seen that for this non-linear method also, *the Gaster transform from spatial to temporal problem, and vice versa, is valid.*

A final word is in order regarding the non-linear term N_1, and N_p in general. Each harmonic level p is A^p in order of magnitude, where A is the amplitude norm for the fundamental. Yet, due to certain difficulties, one cannot make a rational asymptotic expansion in powers of A. At best one can stipulate that the entire N_1 is $\sim O(\alpha_i) \sim O(c_i)$. Nevertheless, one can certainly expect, (and one does get) reasonably rapid numerical convergence as one considers higher and higher harmonics. In fact numerical convergence decides the highest order of p to be retained in the calculations.

This work was supported by grant number 22 (254) /96 EMR-II of the Council of Scientific and Industrial Research (C.S.I.R.), India.

REFERENCES

[1] Bertolotti, F. P., Herbert, Th. and Spalart, P. R., Linear and non-linear stability of the Blasius boundary layer, J. Fluid Mech., vol. 242, pp. 441-474, 1992.

[2] Gastor, M., A note on the relation between temporally increasing and spatially increasing disturbances in hydrodynamic stability. and non-linear stability of the Blasius boundary layer, J. Fluid Mech., vol. 14, pp. 222-224, 1962.

[3] Gastor, M., On the effects of boundary-layer growth on flow stability, J. Fluid Mech.,vol. 66, pp. 465-480, 1974.

[4] Herbert, T., Finite amplitude stability of plane parallel flows, Proc. AGARD Symp. on Laminar-Turbulent Transition, Paper no. 3, AGARD-CP-224, 1977.

[5] Herbert, T., Non-linear stability of parallel flows by higher order amplitude expansions, AIAA Journ. vol. 18, pp. 243-248, 1980.

[6] Klingmann, B. G. B., Boiko, A. V., Westin, K. J. A., Kozlov, V. V. and Alfredson, P. H., Experiments on the stability of Tollmien-Schlichting waves, Eur. J. Mech. B/Fluids, vol. 12(A.4), pp. 493-514, 1993.

[7] Reynolds. W. C. and Potter, M. C., Finite amplitude instability of parallel shear flows, J.Fluid Mech., vol. 27, pp. 465-492, 1967.

[8] Sen, P. K., and Venkateswarlu, D., On the Stability of Plane Poiseuille Flow to Finite-Amplitude Disturbances, considering the Higher-Order Landau Coefficients,J. Fluid Mech., vol. 133, pp. 179-206, 1983.

[9] Sen, P. K., Venkateswarlu, D. and Maji, S., On the Stability of Pipe-Poiseuille Flow to Finite-Amplitude Axisymmetric and non-Axisymmetric Disturbances, J. Fluid Mech., vol. 158, pp. 289-316, 1985.

[10] Sen, P. K. and Thomas, M. D., The stability of laminar boundary layer Flow over a flat plate, Part1: Methods using the adjoint, Proc. 24th Nat. Conf. Of FMFP, Howrah, G67-74, 1997a.

[11] Sen, P. K., Thomas, M. D. and Khurana, R., The stability of laminar boundary layer Flow over a flat plate, Part2: Integral energy methods, Proc. 24th Nat. Conf. Of FMFP, Howrah, G75-82, 1997b.

[12] Sen, P. K. and Vashist, T. K., On the Non-linear Stability of Boundary-Layer Flow over a Flat Plate, Proc. Royal Soc. Lon. A 424, 81-92, 1989.

[13] Stuart, J. T., On the Non-linear mechanisms of hydrodynamic stability, J. Fluid Mech. 4, 1-21, 1957.

[14] Stuart, J. T., On the non-linear mechanisms of wave disturbances in stable and unstable parallel flows, Part 1, The basic behaviour in plane Poiseuille flow, J. Fluid Mech., vol. 9, pp. 353-370, 1960.

[15] Watson, J., On the non-linear mechanisms of wave disturbances in stable and unstable parallel flows, Part 2, The development of a solution for plane Poiseuille flow and for plane Couette flow, J. Fluid Mech., vol. 9, pp. 371-389, 1960.

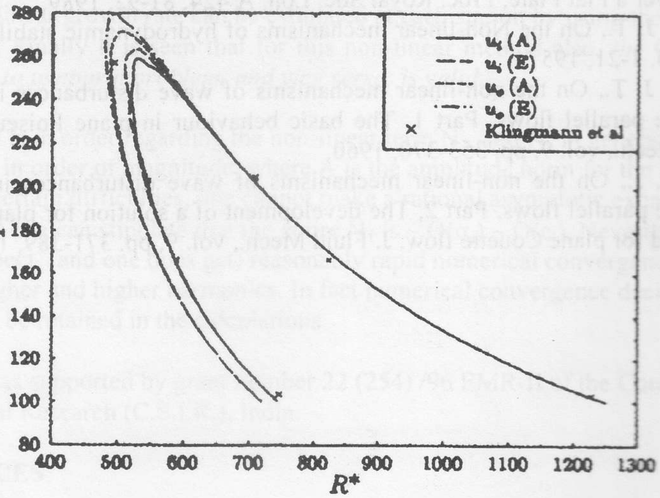

Figure 1. Neutral curves for u_i and u_o for FLS.

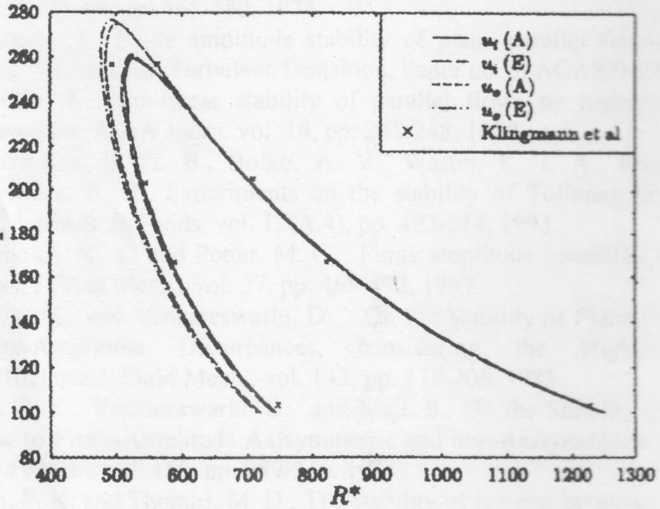

Figure 2. Neutral curves for u_i and u_o for VLS.

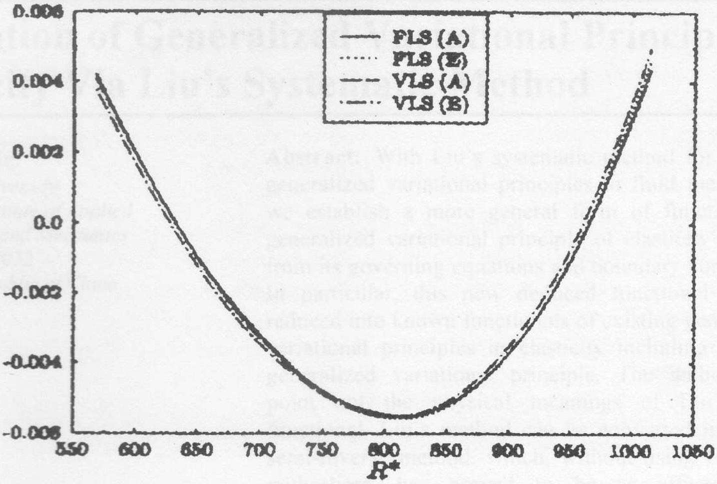

Figure 3. Growth rates for u_i at F=140.

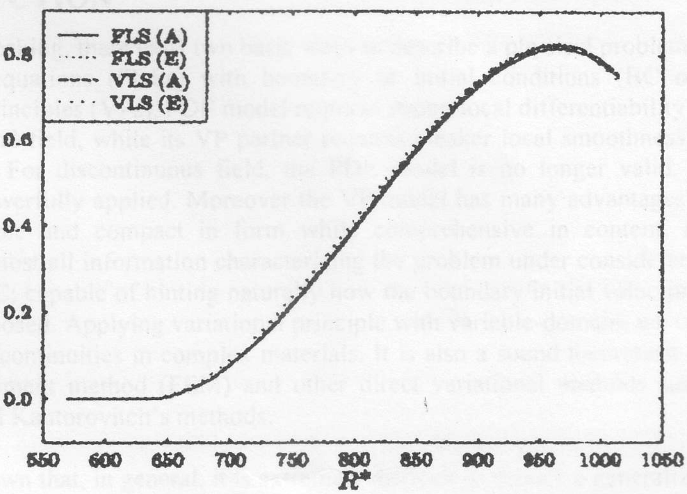

Figure 4. Cumulative growth rates for u_i at F=140.